1. Il semble que f soit continue en $(0,0)$. Prouvons que

$$(1):\ \forall\varepsilon>0,\ \exists\alpha>0,\ \forall V=(x,y),\ \|V\|_\infty\leq\alpha\Rightarrow|f(V)|\leq\varepsilon,$$

   en choisissant la norme $\|.\|_\infty$ sur $\mathbb{R}^2$ (toutes les normes sur $\mathbb{R}^2$ sont équivalentes).
   Soit $\alpha>0$ et $V=(x,y)\in\mathbb{R}^2$ tel que $0<\|V\|_\infty\leq\alpha$.
   $|f(V)|\leq\dfrac{|x|^3|y|}{x^4+y^2}+|x||y|+2|y|$ et $x^4+y^2\geq 2x^2|y|$ (car $(x^2-|y|)^2\geq 0$) donc $|f(V)|\leq\dfrac{|x|}{2}+|x||y|+2|y|\leq\dfrac{5}{2}\alpha+\alpha^2$.
   $\lim\limits_{\alpha\to 0}\dfrac{5}{2}\alpha+\alpha^2=0$ d'où, pour tout $\varepsilon>0$ l'existence de $\alpha>0$ vérifiant (1).

2. $D_{\overrightarrow{n}}f(c)=\lim\limits_{t\to 0}\dfrac{f(c+t\overrightarrow{n}-f(c)}{t}=aD_1f(c)+bD_2f(c)$ et $\overrightarrow{n}$ est unitaire ssi $\overrightarrow{n}=(\cos\theta,\sin\theta)$ pour un θ.

3. f a 3 points critiques, $pc1=(1,0),pc2,pc3$.
   $f(1,y)=-\dfrac{y^3}{1+y^2}$ change de signe pour $y=0$ donc $pc1$ n'est ni un max. ni un min..
   Le terme en r^2 du développement demandé prouve que $f(pc2)$ est un max. local et $f(pc3)$ un min. local.
   $f(x,x)\underset{x\to+\infty}{\sim}x^2\to+\infty$ donc f n'a pas de max. global.
   De même, en étudiant $f(x,-x)$, f n'a pas de min. global.

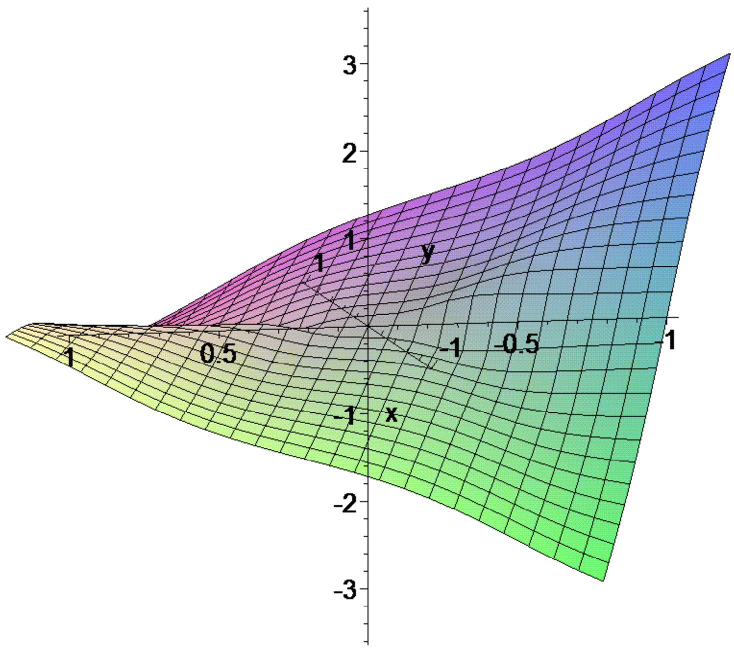
4. $D_1f(0,0)=0$ et $D_2f(0,0)=-2$ en cherchant les limites de taux d'accroissement.
   Elles sont discontinues : en effet $D_1f(x,x^2)\to_{x\to 0}1/2\neq D_1f(0,0)$ et $D_2f(x,0)\to_{x\to 0}\infty$.

O19-C04

```
> restart;
> with(plots):
> f:=x^3*y/(x^4+y^2)+x*y-2*y;
```

$$f := \frac{x^3 y}{x^4+y^2} + x\,y - 2\,y$$

```
> plot3d(f,x=-1..1,y=-1..1);
```




```
> d1:=factor(diff(f,x));d2:=factor(diff(f,y));
```

$$d1 := \frac{y\,(-x^6+3\,x^2\,y^2+x^8+2\,x^4\,y^2+y^4)}{(x^4+y^2)^2}$$

$$d2 := \frac{x^7-x^3\,y^2+x^9+2\,x^5\,y^2+x\,y^4-2\,x^8-4\,x^4\,y^2-2\,y^4}{(x^4+y^2)^2}$$

```
> h:=subs({x=1,y=1},d1*cos(t)+d2*sin(t));
```

$$h := \frac{3}{2}\cos(t) - \sin(t)$$

```
> maximize(h,t=-Pi..Pi,location);
```

$$\frac{\sqrt{13}}{2}, \left\{\left[\left\{t = -\arctan\left(\frac{2}{3}\right)\right\}, \frac{\sqrt{13}}{2}\right]\right\}$$

```
> s:=[solve({d1,d2})];
```

$$s := \left[\{y = 0, x = 1\}, \{y = 0, x = 1\}, \left\{y = \frac{1}{3}\mathrm{RootOf}(3\_Z^2 + 314\,\mathrm{RootOf}(3\_Z^2 - 5\_Z + 1) - 73, label = \_L1), x = \mathrm{RootOf}(3\_Z^2 - 5\_Z + 1)\right\}\right]$$

```
> ss:=[allvalues(s[3])];
```

$$ss := \left[\left\{x = \frac{5}{6} + \frac{\sqrt{13}}{6}, y = \frac{\sqrt{-566 - 157\sqrt{13}}}{9}\right\}, \left\{x = \frac{5}{6} + \frac{\sqrt{13}}{6}, y = -\frac{\sqrt{-566 - 157\sqrt{13}}}{9}\right\}, \left\{x = \frac{5}{6} - \frac{\sqrt{13}}{6}, y = \frac{\sqrt{-566 + 157\sqrt{13}}}{9}\right\}, \left\{x = \frac{5}{6} - \frac{\sqrt{13}}{6}, y = -\frac{\sqrt{-566 + 157\sqrt{13}}}{9}\right\}\right]$$

```
> evalf(ss[3]);
```

$$\{x = 0.2324081207, y = 0.02972096570\}$$

```
> pc1:=s[1];pc2:=ss[3];pc3:=ss[4];
```

$$pc1 := \{y = 0, x = 1\}$$

$$pc2 := \left\{x = \frac{5}{6} - \frac{\sqrt{13}}{6}, y = \frac{\sqrt{-566 + 157\sqrt{13}}}{9}\right\}$$

$$pc3 := \left\{x = \frac{5}{6} - \frac{\sqrt{13}}{6}, y = -\frac{\sqrt{-566 + 157\sqrt{13}}}{9}\right\}$$

```
> g:=subs({x=x+r*a,y=y+r*b},f);
```

$$g := \frac{(x + r\,a)^3 (y + r\,b)}{(x + r\,a)^4 + (y + r\,b)^2} + (x + r\,a)(y + r\,b) - 2\,y - 2\,r\,b$$

```
> gg:=series(g,r,3);
```

$$gg := \left(\frac{x^3\,y}{x^4 + y^2} + x\,y - 2\,y\right) + \left(\frac{x^3\,b + 3\,x^2\,a\,y - \frac{x^3\,y\,(2\,y\,b + 4\,x^3\,a)}{x^4 + y^2}}{x^4 + y^2} + x\,b + a\,y - 2\,b\right) r + \left(a\,b + \frac{3\,x^2\,a\,b + 3\,x\,a^2\,y - \frac{x^3\,y\,(b^2 + 6\,x^2\,a^2)}{x^4 + y^2} - \frac{x^2\,(x^5\,b - x\,b\,y^2 - a\,y\,x^4 + 3\,a\,y^3)(2\,y\,b + 4\,x^3\,a)}{(x^4 + y^2)^2}}{x^4 + y^2}\right.$$

$$r^2 + O(r^3)$$

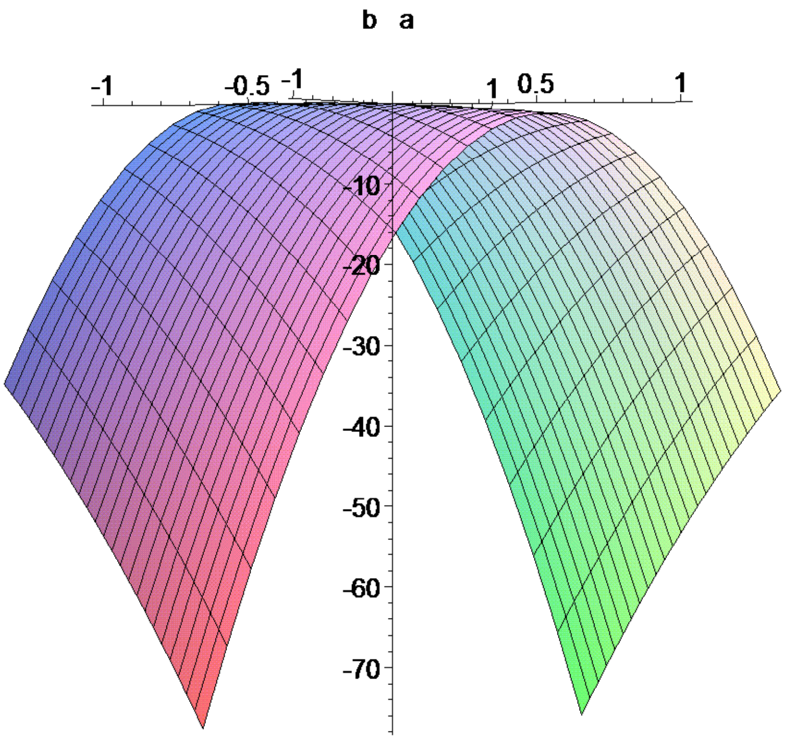
```
> gg1:=subs(pc1,gg);
```

$$gg1 := O(r^3)$$

```
> gg2:=simplify(subs(pc2,gg));
  q2:=coeff(gg2,r,2);
  plot3d(q2,a=-1..1,b=-1..1);
```

$$gg2 := -\frac{2\sqrt{-566+157\sqrt{13}}\,(-343+95\sqrt{13})}{27\,(-263+73\sqrt{13})} - 4\,(642608166\,a\,b - 178227438\,a\,b\sqrt{13} + 10285273\,a^2\sqrt{-566+157\sqrt{13}}\,\sqrt{13} - 37084079\,a^2\sqrt{-566+157\sqrt{13}} + 1548873\sqrt{-566+157\sqrt{13}}\,b^2 - 429579\sqrt{-566+157\sqrt{13}}\,\sqrt{13}\,b^2) \Big/ (9\,(-263+73\sqrt{13})^3)\,r^2 + O(r^3)$$

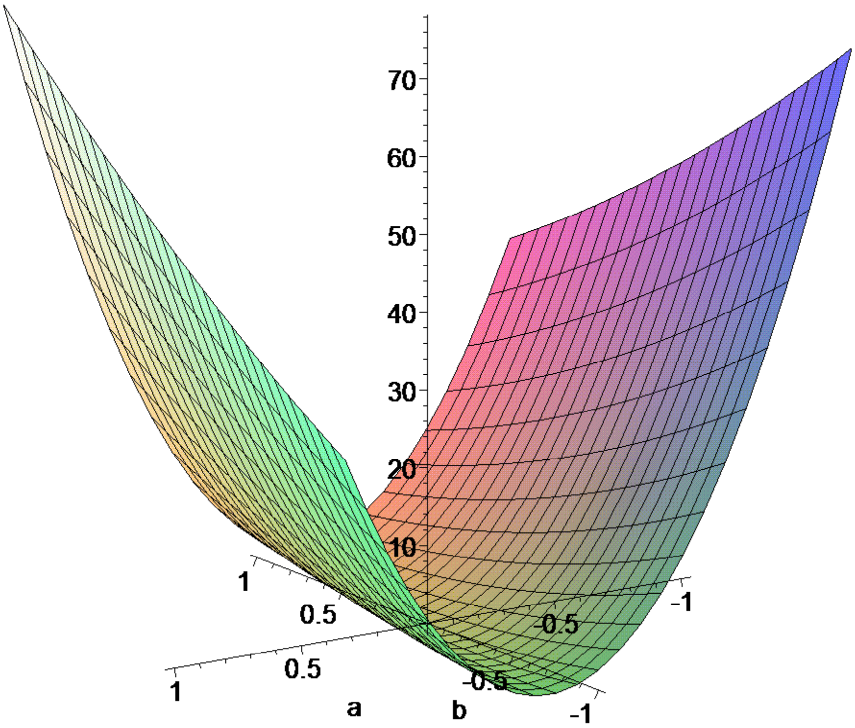
$$q2 := -4\,(642608166\,a\,b - 178227438\,a\,b\sqrt{13} + 10285273\,a^2\sqrt{-566+157\sqrt{13}}\,\sqrt{13} - 37084079\,a^2\sqrt{-566+157\sqrt{13}} + 1548873\sqrt{-566+157\sqrt{13}}\,b^2 - 429579\sqrt{-566+157\sqrt{13}}\,\sqrt{13}\,b^2) \Big/ (9\,(-263+73\sqrt{13})^3)$$




```
> gg3:=simplify(subs(pc3,gg));
  q3:=coeff(gg3,r,2);
  plot3d(q3,a=-1..1,b=-1..1);
```

$$gg3 := \frac{2\sqrt{-566+157\sqrt{13}}\,(-343+95\sqrt{13})}{27\,(-263+73\sqrt{13})} + 4\,(-37084079\,a^2\sqrt{-566+157\sqrt{13}} + 10285273\,a^2\sqrt{-566+157\sqrt{13}}\,\sqrt{13} - 642608166\,a\,b + 178227438\,a\,b\sqrt{13} + 1548873\sqrt{-566+157\sqrt{13}}\,b^2 - 429579\sqrt{-566+157\sqrt{13}}\,\sqrt{13}\,b^2) \Big/ (9\,(-263+73\sqrt{13})^3)\,r^2 + \mathrm{O}(r^3)$$

$$q3 := 4\,(-37084079\,a^2\sqrt{-566+157\sqrt{13}} + 10285273\,a^2\sqrt{-566+157\sqrt{13}}\,\sqrt{13} - 642608166\,a\,b + 178227438\,a\,b\sqrt{13} + 1548873\sqrt{-566+157\sqrt{13}}\,b^2 - 429579\sqrt{-566+157\sqrt{13}}\,\sqrt{13}\,b^2) \Big/ (9\,(-263+73\sqrt{13})^3)$$



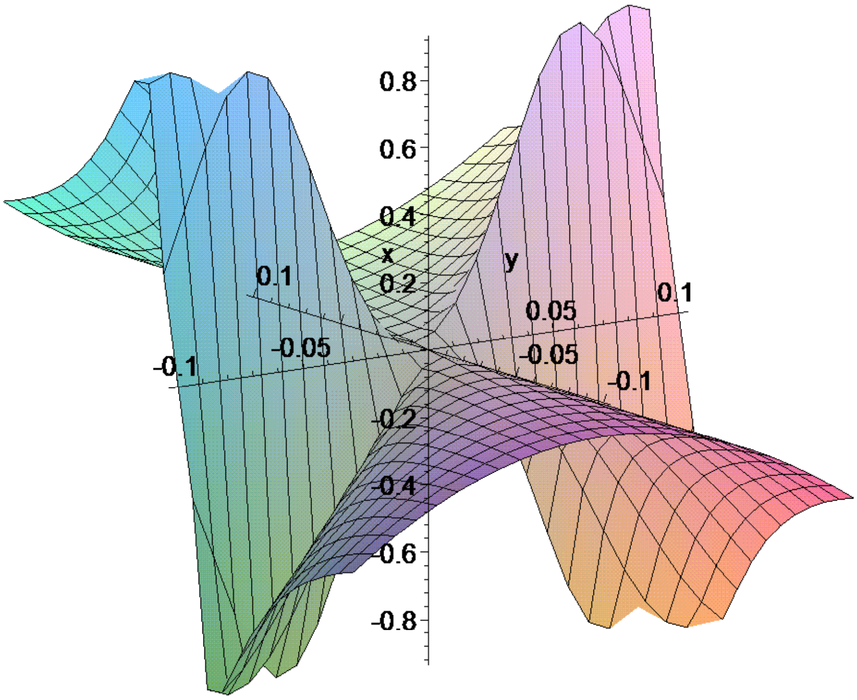

```
> fx:=subs({y=0},f/x);
```

$$fx := 0$$

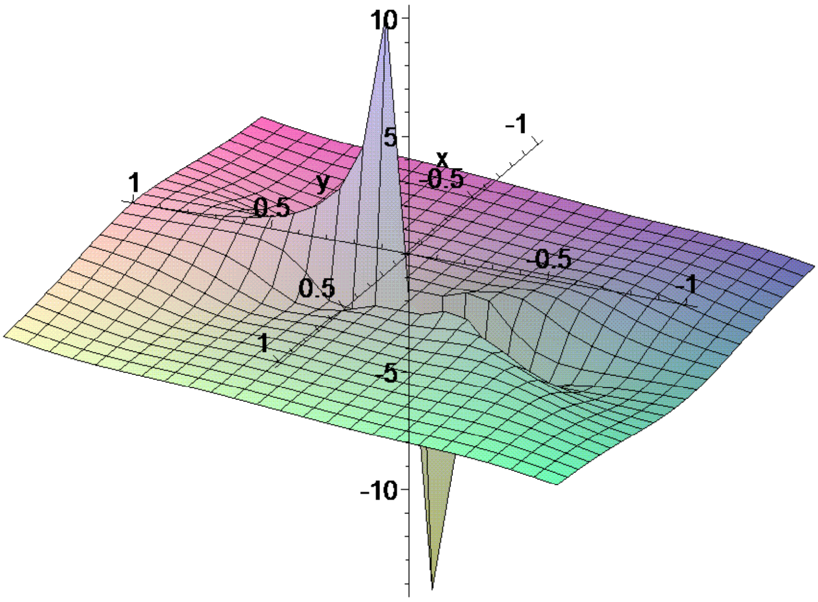
```
> fy:=subs({x=0},f/y);
```

$$fy := -2$$

```
> plot3d(d1,x=-0.1..0.1,y=-0.1..0.1);
```




```
> plot3d(d2,x=-1..1,y=-1..1);
```




```
> subs(y=0,d2);
```

$$\frac{x^7 + x^9 - 2\,x^8}{x^8}$$

```
>
```