a) Il semble que (β_n) soit convergente de limite $L \simeq 0.3678794412$.

b) Par produit de Cauchy : $f(x) = \sum_0^{+\infty} c_n x^n$ où $c_n = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$ pour tout $x \in]-R, R[$ avec $R \geq \min(1, +\infty) = 1$.

D'autre part, $f(x) - xf(x) = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} x^n$ pour tout $x \in \mathbb{R}$ et f admet un DSE sur $]-1, 1[$ (par produit de fonctions admettant des DSE) de la forme $\sum_{n=0}^{+\infty} l_n x^n$.

$\forall x \in]-1, 1[,\ \sum_{n=0}^{+\infty} l_n x^n - \sum_{n=1}^{+\infty} l_{n-1} x^n = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!} x^n$ donc par unicité du DSE :

$$l_0 = 1 \text{ et } \forall n \geq 1,\ l_n - l_{n-1} = \frac{(-1)^n}{n!},$$

et si on pose $\forall n,\ h_n = n! l_n$,

$$h_0 = 1 \text{ et } \forall n \geq 1,\ h_n - n h_{n-1} = (-1)^n$$

donc $\forall n \in \mathbb{N},\ h_n = \alpha_n$ et $l_n = \beta_n$.

c) L'ensemble de définition de f est $\mathbb{R} \setminus \{1\}$ donc le rayon de convergence de sa série de Taylor est $R \leq 1$.
D'après le produit de Cauchy (b)), $R \geq \min(1, +\infty) = 1$, donc $R = 1$.
(c_n) est une suite convergente de limite $1/e \neq 0$ donc la série $\sum c_n x^n$ est divergente (grossièrement) pour $x = -1$.
$f(x) = \sum_0^{+\infty} c_n x^n$ est donc vrai si et seulement si $x \in]-1, 1[$.
Rem : $\forall n,\ c_n = l_n = \beta_n$ donc la conjecture du a) est prouvée et $L = 1/e$.

O19-102

```
> restart:
> suiteA:=proc(n)
  local a,i,aa;
  a:=[1];
  for i from 1 to n do
       aa:=i*a[i]+(-1)^i;
       a:=[op(a),aa]
  od;
  return a;
  end;
```

suiteA := **proc**(n)

local a, i, aa;

a := [1]; **for** i **to** n **do** aa := i*a[i] + (-1)^i; a := [op(a), aa] **end do**; **return** a

end proc

```
> a:=suiteA(31);
```

a := [1, 0, 1, 2, 9, 44, 265, 1854, 14833, 133496, 1334961, 14684570, 176214841, 2290792932, 32071101049, 481066515734, 7697064251745, 130850092279664, 2355301661033953, 44750731559645106, 895014631192902121, 18795307255050944540, 413496759611120779881, 9510425471055777937262, 228250211305338670494289, 5706255282633466762357224, 148362637348470135821287825, 4005791208408693667174771274, 112162153835443422680893595673, 3252702461227859257745914274516, 97581073836835777732377428235481, 3025013288941909109703700275299910]

```
> b:=evalf([seq(a[i+1]/i!,i=2..6),seq(a[10*i+1]/(10*i)!,i=1..3)]);
```

b := [0.5000000000, 0.3333333333, 0.3750000000, 0.3666666667, 0.3680555556, 0.3678794643, 0.3678794412, 0.3678794412]

```
> f:=exp(-x)/(1-x);
```

$$f := \frac{\mathbf{e}^{(-x)}}{1-x}$$

```
> ff:=series(f,x=0,32);
```

$$ff := 1 + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{3}{8}x^4 + \frac{11}{30}x^5 + \frac{53}{144}x^6 + \frac{103}{280}x^7 + \frac{2119}{5760}x^8 + \frac{16687}{45360}x^9 + \frac{16481}{44800}x^{10} + \frac{1468457}{3991680}x^{11} + \frac{16019531}{43545600}x^{12} + \frac{63633137}{172972800}x^{13} + \frac{2467007773}{6706022400}x^{14} + \frac{34361893981}{93405312000}x^{15} + \frac{15549624751}{42268262400}x^{16} + \frac{8178130767479}{22230464256000}x^{17} + \frac{138547156531409}{376610217984000}x^{18} + \frac{92079694567171}{250298560512000}x^{19} + \frac{4282366656425369}{11640679464960000}x^{20} + \frac{72289643288657479}{196503623737344000}x^{21} + \frac{6563440628747948887}{17841281393295360000}x^{22} + \frac{39299278806015611311}{106826515449937920000}x^{23} + \frac{9923922230666898717143}{26976017466662584320000}x^{24} +$$

$$\frac{79253545592131482810517}{215433472824041472000000} x^{25} + \frac{5934505493938805432851513}{16131658445064225423360000} x^{26} +$$
$$\frac{14006262966463963871240459}{38072970106357874688000000} x^{27} + \frac{461572649528573755888451011}{1254684545727217532928000000} x^{28} +$$
$$\frac{116167945043852116348068366947}{315777214062132212662272000000} x^{29} + \frac{3364864615063302680426807870189}{9146650338351415815045120000000} x^{30} +$$
$$\frac{277778998066291010992075323719}{755081602771159120084992000000} x^{31} + O(x^{32})$$

```
> d:=evalf([coeffs(convert(ff,polynom),x)]);
```

d := [1., 0.5000000000, 0.3333333333, 0.3750000000, 0.3666666667, 0.3680555556, 0.3678571429, 0.3678819444, 0.3678791887, 0.3678794643, 0.3678794392, 0.3678794413, 0.3678794412, 0.3678794412, 0.3678794412, 0.3678794412, 0.3678794412, 0.3678794412, 0.3678794412, 0.3678794412, 0.3678794412, 0.3678794412, 0.3678794412, 0.3678794412, 0.3678794412, 0.3678794412, 0.3678794412, 0.3678794412, 0.3678794412, 0.3678794412, 0.3678794412]

```
> dd:=[seq(d[i],i=2..6),seq(d[10*i+1],i=1..3)];
```

dd := [0.5000000000, 0.3333333333, 0.3750000000, 0.3666666667, 0.3680555556, 0.3678794392, 0.3678794412, 0.3678794412]

```
> permut:=proc(l)
  local n,vu,i,p;
  n:=nops(l);
  vu:=[seq(0,i=1..n)];
  for i from 1 to n do
          if l[i]>0 and l[i]<n+1 then
                vu[l[i]]:=1;
          fi;
  od;
  p:=product('vu[i]','i'=1..n);
  if p=0 then
          return false
  else
          return true
  fi;
  end;
```

permut := **proc**(*l*)
local *n*, *vu*, *i*, *p*;
n := nops(*l*);
vu := [seq(0, *i* = 1 .. *n*)];
for *i* **to** *n* **do if** -1 < *l*[*i*] **and** *l*[*i*] < *n* + 1 **then** *vu*[*l*[*i*]] := 1 **end if end do**;
p := product('*vu*[*i*]', '*i*' = 1 .. *n*);
if *p* = 0 **then return** *false* **else return** *true* **end if**
end proc

```
> l:=[1,1,2,3]:permut(l);
```

false

```
> l:=[4,1,5,2,3]:permut(l);
```

true

```
>
```