1. La solution telle que $y(0) = y'(0) = 1$ est $x \mapsto e^x e^{x^2/2}$.
2. Il semblerait que tous les $y^{(k)}(0)$, $k \in \mathbb{N}$ sont des entiers de $\mathbb{N}$.
3. f est une involution de $[\![1, n+2]\!]$ ssi $f(n+2) = n+2$ et $f_{|[\![1,n+1]\!]}$ est une involution de $[\![1, n+1]\!]$ ou bien $\exists ! k \in [\![1, n+1]\!]$, $f(n+2) = k$ et $f(k) = n+2$ et $f_{|[\![1,n+1]\!] \setminus \{k\}}$ est une involution de $[\![1, n+1]\!] \setminus \{k\}$, les 2 cas sont incompatibles et le nombre d'involutions de chaque $[\![1, n+1]\!] \setminus \{k\}$ est I_n donc $I_{n+2} = I_{n+1} + \sum_{k=1}^{n+1} I_n$ CQFD.
4. Toute involution de $[\![1, n]\!]$ est une permutation donc $\forall n \in \mathbb{N},\ I_n \leq n!$ d'où $R \geq 1$.
5. L'EDO donnée peut aussi s'écrire $y' = (1+x)y + cste$, la constante étant nulle pour la fonction $x \mapsto e^x e^{x^2/2}$. La suite $a_n = I_n/n!$ vérifie la relation de récurrence $(n+2)a_{n+2} = a_n + a_{n+1}$ et c'est aussi ce qu'on trouve en résolvant $y' = (1+x)y$ par séries entières. De plus $I_0 = 1$ d'où l'égalité.

O18-092

```
> restart;
> with(DEtools);
```

[*AreSimilar, DEnormal, DEplot, DEplot3d, DEplot_polygon, DFactor, DFactorLCLM, DFactorsols, Dchangevar, FunctionDecomposition, GCRD, Gosper, Heunsols, Homomorphisms, IsHyperexponential, LCLM, MeijerGsols, MultiplicativeDecomposition, ODEInvariants, PDEchangecoords, PolynomialNormalForm, RationalCanonicalForm, ReduceHyperexp, RiemannPsols, Xchange, Xcommutator, Xgauge, Zeilberger, abelsol, adjoint, autonomous, bernoullisol, buildsol, buildsym, canoni, caseplot, casesplit, checkrank, chinisol, clairautsol, constcoeffsols, convertAlg, convertsys, dalembertsol, dcoeffs, de2diffop, dfieldplot, diff_table, diffop2de, dperiodic_sols, dpolyform, dsubs, eigenring, endomorphism_charpoly, equinv, eta_k, eulersols, exactsol, expsols, exterior_power, firint, firtest, formal_sol, gen_exp, generate_ic, genhomosol, gensys, hamilton_eqs, hypergeomsols, hyperode, indicialeq, infgen, initialdata, integrate_sols, intfactor, invariants, kovacicsols, leftdivision, liesol, line_int, linearsol, matrixDE, matrix_riccati, maxdimsystems, moser_reduce, muchange, mult, mutest, newton_polygon, normalG2, ode_int_y, ode_y1, odeadvisor, odepde, parametricsol, particularsol, phaseportrait, poincare, polysols, power_equivalent, rational_equivalent, ratsols, redode, reduceOrder, reduce_order, regular_parts, regularsp, remove_RootOf, riccati_system, riccatisol, rifread, rifsimp, rightdivision, rtaylor, separablesol, singularities, solve_group, super_reduce, symgen, symmetric_power, symmetric_product, symtest, transinv, translate, untranslate, varparam, zoom*]

```
> edo:=diff(y(x),x$2)=(1+x)*diff(y(x),x)+y(x);
```

$$edo := \frac{d^2}{dx^2}\,y(x) = (1+x)\left(\frac{d}{dx}\,y(x)\right) + y(x)$$

```
> sol0:=dsolve({edo,y(0)=1, D(y)(0)=1},y(x));
```

$$sol0 := y(x) = \frac{1}{\mathbf{e}^{(-x-1/2\,x^2)}}$$

```
> sol:=seq(subs(dsolve({edo,y(0)=1, D(y)(0)=m},y(x)),y(x)),m=0..5);
```

$$sol := -\frac{1}{2}\,\frac{\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}+\frac{\sqrt{2}}{2}\right)\sqrt{2}\,\sqrt{\pi}\,\mathbf{e}^{(1/2)}}{\mathbf{e}^{(-x-1/2\,x^2)}} + \frac{1}{2}\,\frac{\left(\sqrt{2}+\operatorname{erf}\left(\frac{\sqrt{2}}{2}\right)\sqrt{\pi}\,\mathbf{e}^{(1/2)}\right)\sqrt{2}}{\mathbf{e}^{(-x-1/2\,x^2)}},\ \frac{1}{\mathbf{e}^{(-x-1/2\,x^2)}},$$

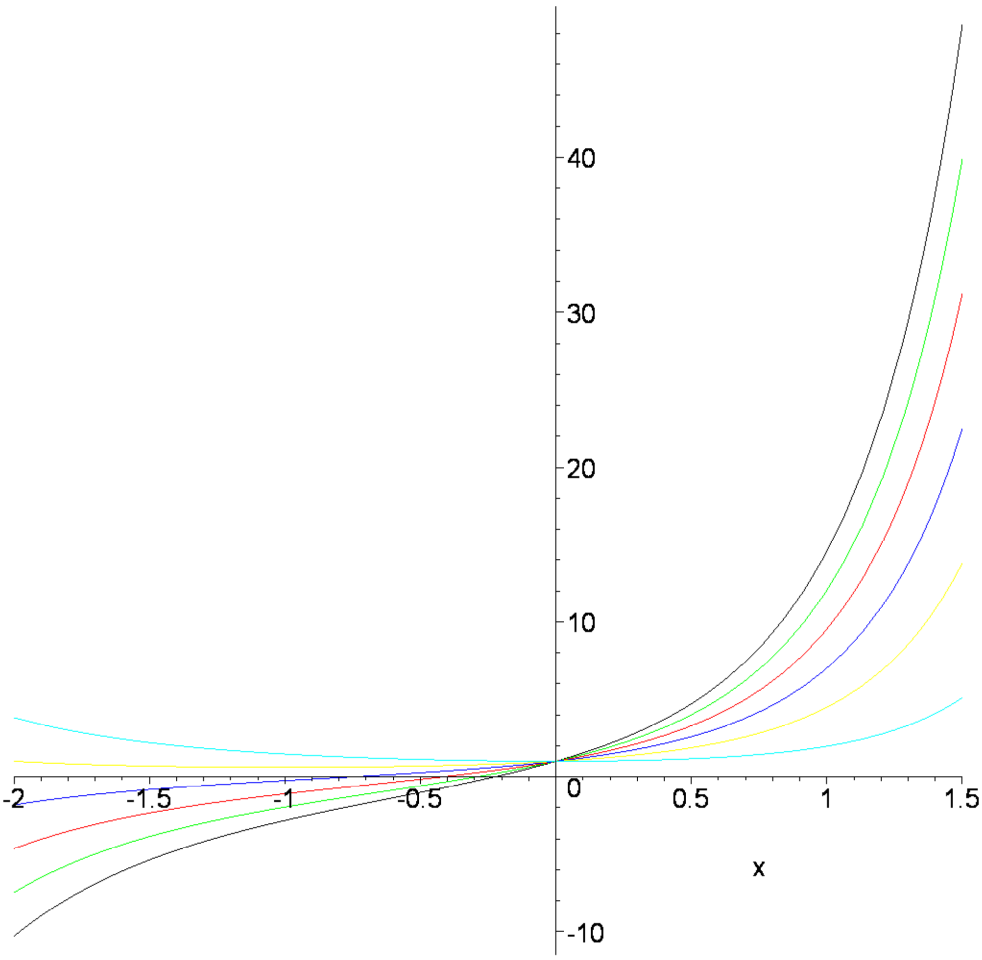
$$\frac{1}{2}\,\frac{\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}+\frac{\sqrt{2}}{2}\right)\sqrt{2}\,\sqrt{\pi}\,\mathbf{e}^{(1/2)}}{\mathbf{e}^{(-x-1/2\,x^2)}} - \frac{1}{2}\,\frac{\left(-\sqrt{2}+\operatorname{erf}\left(\frac{\sqrt{2}}{2}\right)\sqrt{\pi}\,\mathbf{e}^{(1/2)}\right)\sqrt{2}}{\mathbf{e}^{(-x-1/2\,x^2)}},$$

$$\frac{\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}+\frac{\sqrt{2}}{2}\right)\sqrt{2}\,\sqrt{\pi}\,\mathbf{e}^{(1/2)}}{\mathbf{e}^{(-x-1/2\,x^2)}} - \frac{1}{2}\,\frac{\left(-\sqrt{2}+2\operatorname{erf}\left(\frac{\sqrt{2}}{2}\right)\sqrt{\pi}\,\mathbf{e}^{(1/2)}\right)\sqrt{2}}{\mathbf{e}^{(-x-1/2\,x^2)}},$$

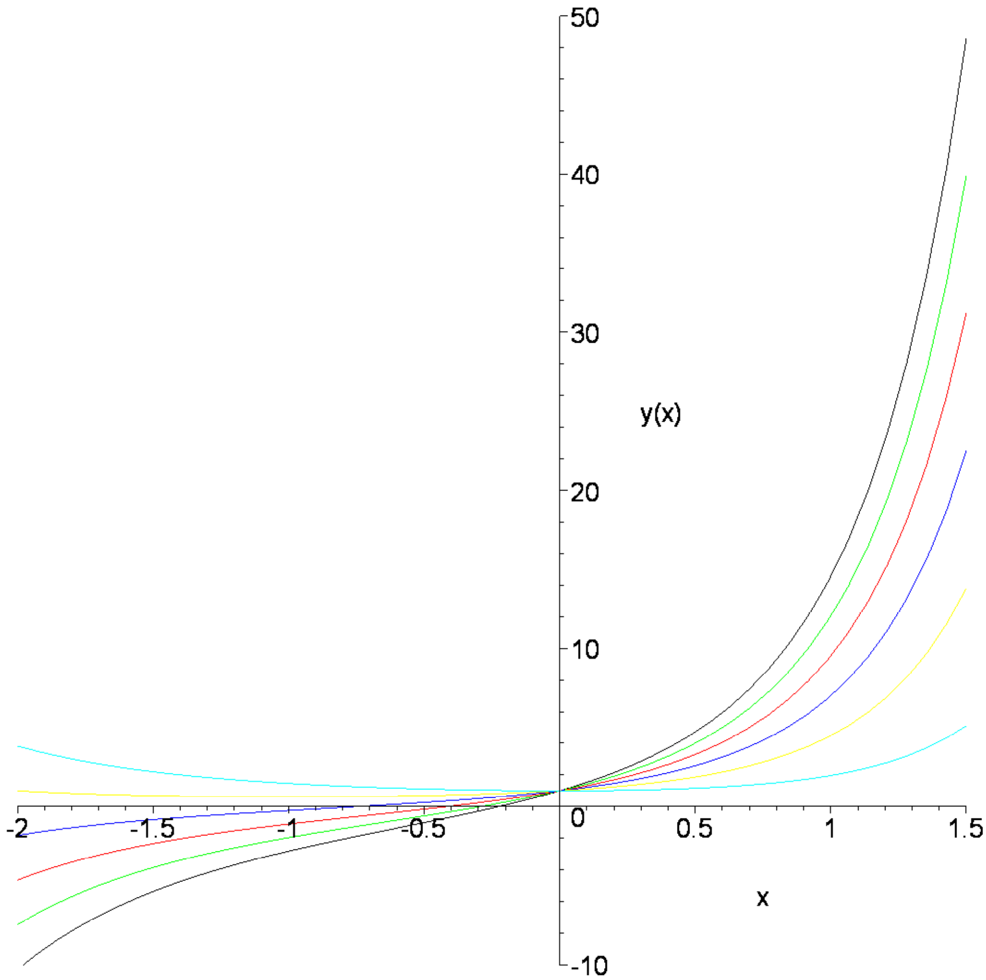
$$\frac{3}{2}\,\frac{\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}+\frac{\sqrt{2}}{2}\right)\sqrt{2}\,\sqrt{\pi}\,\mathbf{e}^{(1/2)}}{\mathbf{e}^{(-x-1/2\,x^2)}} - \frac{1}{2}\,\frac{\left(-\sqrt{2}+3\operatorname{erf}\left(\frac{\sqrt{2}}{2}\right)\sqrt{\pi}\,\mathbf{e}^{(1/2)}\right)\sqrt{2}}{\mathbf{e}^{(-x-1/2\,x^2)}},$$

$$\frac{2\operatorname{erf}\left(\frac{\sqrt{2}\,x}{2}+\frac{\sqrt{2}}{2}\right)\sqrt{2}\,\sqrt{\pi}\,\mathbf{e}^{(1/2)}}{\mathbf{e}^{(-x-1/2\,x^2)}} - \frac{1}{2}\,\frac{\left(-\sqrt{2}+4\operatorname{erf}\left(\frac{\sqrt{2}}{2}\right)\sqrt{\pi}\,\mathbf{e}^{(1/2)}\right)\sqrt{2}}{\mathbf{e}^{(-x-1/2\,x^2)}}$$

```
> plot([sol],x=-2..1.5,color=[cyan,yellow,blue,red,green,black]);
```




```
> DEplot( edo, y(x), x=-2..1.5, y=-10..50, [seq([y(0)=1,D(y)(0)=m],m=0..5)]
  ,linecolor=[cyan,yellow,blue,red,green,black],thickness=1);
```




```
> y0:=subs(sol0,y(x));
```

$$y0 := \frac{1}{e^{(-x-1/2\,x^2)}}$$

```
> taylor(y0,x=0,16);
```

$$1 + x + x^2 + \frac{2}{3}x^3 + \frac{5}{12}x^4 + \frac{13}{60}x^5 + \frac{19}{180}x^6 + \frac{29}{630}x^7 + \frac{191}{10080}x^8 + \frac{131}{18144}x^9 + \frac{1187}{453600}x^{10} + \frac{2231}{2494800}x^{11} + \frac{17519}{59875200}x^{12} + \frac{71063}{778377600}x^{13} + \frac{29881}{1089728640}x^{14} + \frac{323423}{40864824000}x^{15} + O(x^{16})$$

```
> invol:=proc(n)
  local k,i;
  i[0]:=1;
  i[1]:=1;
  for k from 2 to n do
        i[k]:=i[k-1]+(k-1)*i[k-2]
  od;
  return i;
  end;
```

invol :=

proc(n) **local** k, i; $i[0] := 1$; $i[1] := 1$; **for** k **from** 2 **to** n **do** $i[k] := i[k-1] + (k-1)*i[k-2]$ **end do**; **return** i **end proc**

```
> [seq(invol(15)[k],k=0..15)];
```

[1, 1, 2, 4, 10, 26, 76, 232, 764, 2620, 9496, 35696, 140152, 568504, 2390480, 10349536]

```
> i_bis:=n->n!*sum(1/(n-2*q)!/2^q/q!,q=0..floor(n/2));
```

$$i\_bis := n \to n!\left(\sum_{q=0}^{\mathrm{floor}(1/2\,n)} \frac{1}{(n-2\,q)!\,2^q\,q!}\right)$$

```
> [seq(i_bis(k),k=0..15)];
```

[1, 1, 2, 4, 10, 26, 76, 232, 764, 2620, 9496, 35696, 140152, 568504, 2390480, 10349536]

>