1. f est définie sur $\mathbb{R}_+^*$ et y est continue et strictement croissante ; $\lim_{0^+} f = -\infty$ et $\lim_{+\infty} f = +\infty$ donc f est une bijection bicontinue de $\mathbb{R}_+^*$ sur $\mathbb{R}$ d'où l'existence-unicité de la suite (x_n).
   $x \mapsto \dfrac{x}{2} - \ln x$ admet un minimum pour $x = 2$ de valeur $1 - \ln 2 > 0$ donc $\forall n,\ \dfrac{3x_n}{2} \geq x_n + \ln x_n = n \geq x_n$ donc $\forall n,\ \dfrac{2n}{3} \leq x_n \leq n$.
   En particulier, $\lim x_n = +\infty$ donc $n = x_n + \ln x_n \sim x_n$: $x_n = n + o(n)$.
   Soit $y_n = x_n - n (= o(n))$. $y_n + n + \ln(y_n + n) = n$ donc $y_n = -\ln n + \ln(1 + \dfrac{y_n}{n}) = -\ln n + o(1)$ et $x_n = n - \ln n + o(1)$.
   Soit $z_n = x_n - n + \ln n (= o(1))$. $z_n = \ln\left(1 - \dfrac{\ln n}{n} + \dfrac{z_n}{n}\right) \sim -\dfrac{\ln n}{n} + \dfrac{z_n}{n} = -\dfrac{\ln n}{n} + o(1/n)$ d'où $x_n = n - \ln n - \dfrac{\ln n}{n} + o(1/n)$.
2. $(1 + n^p)^{1/p} = n\left(1 + \dfrac{1}{n^p}\right)^{1/p}$ et, si $p > 0$, alors $\dfrac{1}{n^p} \to_{n\to+\infty} 0$ donc, en posant $t = \dfrac{1}{n}$:
   $u_n = \sin\left(n\pi(1 + a_1 t^p + a_2 t^{2p} + o(t^{2p}))\right)$ où a_1, a_2 ne dépendent pas de n ($a_1 = 1/p$).
   $u_n = (-1)^n \sin\left(\pi a_1 t^{p-1} + \pi a_2 t^{2p-1} + o(t^{2p-1})\right)$.
   Supposons $p > 1$: on peut écrire un DL de sin : $u_n = (-1)^n \left(u + b_3 u^3 + ... + b_h u^h + o(u^h)\right)$ où $b_3 ... b_h$ ne dépendent pas de n et $u = \pi a_1 t^{p-1} + \pi a_2 t^{2p-1} + o(t^{2p-1}) \underset{\pi}{\sim} a_1 t^{p-1}$ donc, si $h \geq \dfrac{2p-1}{p-1}$, alors $u^h = O(t^{2p-1})$ donc :
   $u_n = \displaystyle\sum_{1\leq i\leq 2, 1\leq j\leq h} (-1)^n c_{i,j} t^{(ip-1)j} + o(t^{2p-1})$, or les $(-1)^n t^{(ip-1)j} = \dfrac{(-1)^n}{n^{(ip-1)j}}$ sont des TG de séries alternées convergentes, et $2p - 1 > 1$ donc $\sum o(t^{2p-1})$ est absolument convergente, d'où : $\sum u_n$ converge.
   Remarques :
   - Si $p > 2$, on a tout simplement $u_n = (-1)^n \dfrac{1}{p} \dfrac{1}{n^{p-1}} + o(1/n^2)$. Par contre, si $p > 1$ est proche de 1, il faudra un "long" développement de sin, par ex. pour $p = 1.1$:
     $u_n = (-1)^n \left(\left[\dfrac{1}{p}\dfrac{1}{n^{0.1}} + \dfrac{1}{2p}(\dfrac{1}{p} - 1)\dfrac{1}{n^{1.2}}\right] - \dfrac{1}{6}\left[\dfrac{1}{p}\dfrac{1}{n^{0.1}} + \dfrac{1}{2p}(\dfrac{1}{p} - 1)\dfrac{1}{n^{1.2}}\right]^3 ... + b_{11}\left[\dfrac{1}{p}\dfrac{1}{n^{0.1}} + \dfrac{1}{2p}(\dfrac{1}{p} - 1)\dfrac{1}{n^{1.2}}\right]^{11} + o(1/n^{1.1}) \right)$.
   - Si $0 < p < 1$, on a le sin d'un angle de limite infinie.
   - Si $p = 1$, le calcul direct donne $u_n = (-1)^{n+1}$ d'où la divergence.

018 − 079

```
> restart;
> eq := x + ln(x) − n;
```

$$eq := x + \ln(x) - n \quad (1)$$

```
> u := unapply(solve(eq, x), n);
```

$$u := n \rightarrow e^{-\mathrm{LambertW}(e^{n}) + n} \quad (2)$$

```
> map(evalf, [seq(u(10· n), n = 1 ..5)]);
```

$$[7.929420095, 17.15756098, 26.71478293, 36.40528586, 46.16771896] \quad (3)$$

```
> map(evalf, [seq(u(10· n) − 10· n, n = 1 ..5)]);
```

$$[-2.070579905, -2.84243902, -3.28521707, -3.59471414, -3.83228104] \quad (4)$$

```
> map(evalf, [seq(u(10· n) − 10·n + ln(10·n), n = 1 ..5)]);
```

$$[0.232005188, 0.153293254, 0.115980312, 0.094165314, 0.079741965] \quad (5)$$

```
> map(evalf, [seq(u(10· n) − 10·n + ln(10·n) − ln(10·n)/(10·n), n = 1 ..5)]);
```

$$[0.001746679, 0.003506640, 0.002607066, 0.001943328, 0.001501505] \quad (6)$$

```
> asympt(u(n), n);
```

$$n - \ln(n) + \frac{-\frac{1}{2}\ln(n)(\ln(n)-2) + \frac{1}{2}\ln(n)^2}{n} \quad (7)$$
$$+ \frac{-\frac{1}{6}\ln(n)\left(-9\ln(n)+6+2\ln(n)^2\right) + \frac{1}{2}\ln(n)^2(\ln(n)-2) - \frac{1}{6}\ln(n)^3}{n^2}$$
$$+ \frac{1}{n^3}\Big(-\frac{1}{12}\ln(n)\left(3\ln(n)^3-22\ln(n)^2+36\ln(n)-12\right) + \frac{1}{6}\ln(n)^2\left(-9\ln(n)+6\right.$$
$$\left.+2\ln(n)^2\right) + \frac{1}{8}\ln(n)^2(\ln(n)-2)^2 - \frac{1}{4}\ln(n)^3(\ln(n)-2) + \frac{1}{24}\ln(n)^4\Big)$$
$$+ \frac{1}{n^4}\Big(-\frac{1}{60}\ln(n)\left(-125\ln(n)^3+350\ln(n)^2+12\ln(n)^4-300\ln(n)+60\right)$$
$$+ \frac{1}{12}\ln(n)^2\left(3\ln(n)^3-22\ln(n)^2+36\ln(n)-12\right) + \frac{1}{12}\ln(n)^2(\ln(n)-2)\big($$
$$-9\ln(n)+6+2\ln(n)^2\big) - \frac{1}{120}\ln(n)^5 - \frac{1}{3}\ln(n)\Big(\frac{1}{6}\ln(n)^2\left(-9\ln(n)+6\right.$$
$$\left.+2\ln(n)^2\right) + \frac{1}{8}\ln(n)^2(\ln(n)-2)^2\Big) - \frac{1}{36}\ln(n)^3\left(-9\ln(n)+6+2\ln(n)^2\right)$$
$$- \frac{1}{12}\ln(n)^3(\ln(n)-2)^2 + \frac{1}{12}\ln(n)^4(\ln(n)-2)\Big) + O\left(\frac{1}{n^5}\right)$$

```
> restart;
> t := (n, p) → (1 + n^p)^(1/p);
```

(8)

$$t := (n, p) \rightarrow \left(1 + n^p\right)^{\frac{1}{p}} \tag{8}$$

```
> u := (n, p) → sin(π·(1 + n^p)^(1/p));
```

$$u := (n, p) \rightarrow \sin\left(\pi \left(1 + n^p\right)^{\frac{1}{p}}\right) \tag{9}$$

```
> s := (p, k) → sum(u(p), n = 1 ..k);
```

$$s := (p, k) \rightarrow \sum_{n=1}^{k} u(p) \tag{10}$$

```
> SPart :=proc(n, p)
   local l, k;
   l := [u(1, p)];
   for k from 2 to 10· n do
         l := [op(l), l[k − 1] + u(k, p)]
   od;
   return evalf([seq(l[10·k], k = 1 ..n)]);
   end;
```

(11)

```
SPart := proc(n, p)
   local l, k;
   l := [u(1, p)];
   for k from 2 to 10 * n do l := [op(l), l[k − 1] + u(k, p)] end do;
   return evalf([seq(l[10 * k], k = 1 ..n)])
end proc
```

```
> SPart(40, 2);
```

[-0.4923894691, -0.5283523397, -0.5408561921, -0.5472000105, -0.5510347291, -0.5536029493, -0.5554431809, -0.5568263463, -0.5579040310, -0.5587673820, -0.5594748708, -0.5600641841, -0.5605633125, -0.5609917519, -0.5613633979, -0.5616885493, -0.5619756046, -0.5622304622, -0.5624585220, -0.5626645832, -0.5628501456, -0.5630195090, -0.5631737730, -0.5633154380, -0.5634459035, -0.5635662693, -0.5636778351, -0.5637811011, -0.5638780677, -0.5639677343, -0.5640520011, -0.5641308685, -0.5642041355, -0.5642754025, -0.5643426698, -0.5644049372, -0.5644642046, -0.5645194720, -0.5645717394, -0.5646220070] (12)

```
> SPart(40, 3);
```

[-0.5496589685, -0.5531313995, -0.5538126563, -0.5540558942, -0.5541697622, -0.5542321302, -0.5542697980, -0.5542944660, -0.5543114340, -0.5543236018, -0.5543327699, -0.5543399379, -0.5543453058, -0.5543496738, -0.5543534420, -0.5543560097, -0.5543585775, -0.5543600455, -0.5543618135, -0.5543631814, -0.5543642493, -0.5543653173, -0.5543666852, -0.5543665532, -0.5543672212, -0.5543679890, -0.5543685570, -0.5543691250, -0.5543696929, -0.5543699609, (13)

-0.5543702289, -0.5543703967, -0.5543706647, -0.5543719326, -0.5543722006, -0.5543714684, -0.5543707364, -0.5543710042, -0.5543712722, -0.5543725402]

```
> SPart(40, 1.3);
```

[0.1221850640, 0.0721871431, 0.0392883268, 0.0156002718, -0.0026220882, -0.0172827652, -0.0294602852, -0.0398186822, -0.0487930302, -0.0566826902, -0.0637023212, -0.0700092722, -0.0757232832, -0.0809374572, -0.0857250872, -0.0901440462, -0.0942426132, -0.0980606332, -0.1016299322, -0.1049775962, -0.1081272182, -0.1110993822, -0.1139105682, -0.1165755302, -0.1191090582, -0.1215210162, -0.1238205772, -0.1260179882, -0.1281211172, -0.1301364262, -0.1320700102, -0.1339283352, -0.1357170272, -0.1374367652, -0.1390953992, -0.1406971652, -0.1422454352, -0.1437408762, -0.1451896642, -0.1465906682] **(14)**

```
> SPart(40, 1.2);
```

[0.2422640768, 0.2379105648, 0.2273444215, 0.2171115835, 0.2079232135, 0.1997462095, 0.1924373755, 0.1858561775, 0.1798851535, 0.1744294175, 0.1694130195, 0.1647748295, 0.1604645755, 0.1564415175, 0.1526710795, 0.1491254315, 0.1457808495, 0.1426163775, 0.1396147095, 0.1367606195, 0.1340411215, 0.1314446775, 0.1289608015, 0.1265809615, 0.1242973015, 0.1221027495, 0.1199902755, 0.1179550015, 0.1159912315, 0.1140947655, 0.1122612675, 0.1104869135, 0.1087670735, 0.1071009495, 0.1054847175, 0.1039158415, 0.1023888615, 0.1009044995, 0.0994606375, 0.0980543555] **(15)**

```
> SPart(40, 0.2);
```

[0.0701223067, 4.996884133, 5.051221568, 5.121316356, 5.513728712, 5.845863667, 6.756691407, -0.5027196277, -6.644382717, -5.675903571, -6.130684503, -8.154888425, -7.997396453, -7.966857433, -6.571658007, -7.647698321, -7.854649761, -7.600045019, -6.714529735, -7.777106117, -6.804618631, -6.834603075, -6.937156058, -6.766677180, -7.057038280, -7.757170892, -6.759182588, -7.676278043, -7.386716281, -6.831701419, -6.768435679, -6.776560708, -6.778094798, -7.111294038, -7.736029160, -7.203115640, -7.018690474, -7.577469866, -7.072693036, -7.097087526] **(16)**

```
> map(evalf, [seq(s(0.2, 10·k), k = 20 ..30)]);
```

[-7.777106132, -6.804618654, -6.834603074, -6.937156066, -6.766677202, -7.057038280, -7.757170898, -6.759182574, -7.676278028, -7.386716250, -6.831701386] **(17)**

```
> map(evalf, [seq(u(k, 0.2), k = 10 ..40)]);
```

[-0.9528979393, 0.9215356843, -0.4614795107, -0.6771558046, 0.6445163015, 0.9891187999, 0.7113420404, 0.4900203764, 0.5279527368, 0.7823625801, 0.9985486228, 0.6811646579, -0.3576252096, -0.9909674060, 0.1139063065, 0.9575593951, -0.6116188441, -0.3611911586, 0.9421336463, -0.9367182293, 0.6176942775, -0.2605599624, 0., 0.1306553921, -0.1346672459, 0.01733930487, 0.2156402847, -0.5349669785, 0.8520030857, -0.9999630961, 0.7846140014] **(18)**